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§1 Matrices

Definition 1.1 Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ its Jacobian is a $m \times n$ matrix given by,

$$\mathbf{J}_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

The Jacobian of a composition of functions is given by the chain rule if $H(x) = F(G(x))$, then $\mathbf{J}_H = \mathbf{J}_F(G(x))\mathbf{J}_G$

In order to find the *column space* of A (søylerommet) one needs to *row-reduce* the matrix and find the *pivot* columns (those with 1), then select all columns in the original matrix with *pivots* in a set S. The columns of S are a basis for A.

The Hessian matrix of a function is given by,

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Then one can apply a very nice lemma,

Lemma 1.2 Let $D = \det \mathbf{H}_f$, then if $D < 0$ the point is a *saddle* point, if greater than 0 it is an extrema, otherwise we cannot say anything.

§2 Dot Product Properties

- If $u, v \in \mathbb{C}^n$ then
$$u \cdot v = v^T \overline{u}$$
where $\overline{u}$ is the complex conjugate of the vector.
- $a \cdot b = |a||b| \cos \theta$
- The area of a parallelogram given by the vectors a, b is $|a \times b|$ which is also a determinant of a 3×3 matrix.
- The volume of a parallelepiped over a, b, c is given by $|a \cdot (b \times c)|$ and the volume of a tetrahedron is given by that of the parallelepiped over 6.

§3 Eigen Values/Vectors

Definition 3.1 Given a matrix M an eigenvector is a vector $\mathbf{v}$ such that $M\mathbf{v} = \lambda \mathbf{v}$, similarly the value of $\lambda \in \mathbb{R}$ is the eigenvalue.

Lemma 3.2 Any symmetric $n \times n$ matrix M has real eigenvalues.

§4 Integral Techniques

- For any closed curve C and scalar function f defined over it,
$$\int_C f ds = \int_a^b f(\mathbf{r}(t))\mathbf{r}'(t)dt$$
- Given a curve defined by $r(t) = \Big(f(t), g(t), h(t)\Big)$ with $a \leq t \leq b$ the length of the curve can be calculated as,
$$\mathcal{L}(C) = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt$$
- Given some vector field $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$ along with some parameterized curve C given by $r(t)$ with $a \leq t \leq b$ we can calculate,
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

- Given an open subset $A \subseteq \mathbb{R}^m$ let $G : (\vec{x}, t) \to \mathbb{R}$ be some continuous function with continuous partial derivatives, then let us define a function,

$$F(\vec{x}) = \int_a^b F(\vec{x}, t) dt$$

then it must be that,

$$\frac{\partial F(\vec{x})}{\partial x_i} = \int_a^b \frac{\partial F(\vec{x}, t)}{\partial x_i} dt$$

- Given some transformation $T : (x, y) \to (u, v)$, then,

$$\iint\limits_S f(x, y) dx dy = \iint\limits_S f(T(x, y)) |\det \mathbf{J}_T| \, du dv$$

When doing polar coordinates substitution in 2d, the factor simplifies to r.

Remark 4.1 To efficiently find values try to find the *intersection plane* then integrate with that plane as a *base*.

§5 Convergence Results

Definition 5.1 A sequence of points $x_1, x_2, \ldots$ *converges* to x if for all $\epsilon > 0$ there exists an N such that for all $i \geq N$ we have that,

$$|x_i - x| < \epsilon$$

Definition 5.2 A sequence $a_1, a_2, \ldots$ is a *Cauchy sequence* if for all $\epsilon > 0$ there exists an N such that for all $n, m \geq N$,

$$|a_n - a_m| < \epsilon$$

Importantly it turns out that a sequence is *Cauchy* if and only if it is *convergent*.

Theorem 5.3 **(Bolzano-Weierstrass' theorem)** Every bounded sequence in $\mathbb{R}^n$ has a *convergent subsequence*.

Definition 5.4 A function $f : A \to \mathbb{R}^n$ is *uniformly continuous* if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all vectors $\mathbf{u}, \mathbf{v} \in A$ such that $|\mathbf{u} - \mathbf{v}| < \delta$ it must be that $|f(\mathbf{u}) - f(\mathbf{v})| < \epsilon$.

Theorem 5.5 **(Heine–Cantor theorem)** If $f : A \to \mathbb{R}^n$ is *continuous* and A is compact (closed and bounded), then f is *uniformly continuous*.

Contractions

Definition 5.6 A function $F : A \to A$ is called a *contraction* if there exists some $C < 1$ such that,

$$|\vec{F}(x) - \vec{F}(y)| < C|x - y|$$

for all $x, y \in A$ and C is called the *contraction factor*.

- If F is a C-*contraction*, then $F^{\circ n}$ is also a *contraction* with coefficient C^n.
- **(Banach Fixed Point Theorem)** If $F : A \to A$ is a *contraction* over a closed subset of $\mathbb{R}^n$, then F has a unique fixed point. Any sequence $x_i = F(x_{i-1})$ with any starting point converges to that unique fixed point.

§6 Green's Theorem

If C is the boundary of some region D, then,

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

We can apply this theorem to find areas, for example we know that,

$$A = \iint_D dA$$

thus we only need $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and we will be able to apply Green's theorem to evaluate A. Below I take $Q = x, 0, \frac{x}{2}$ and $P = 0, -y, \frac{y}{2}$,

$$A = \int_C xdy = -\int_C ydx = \frac{1}{2}\int_c xdy + ydx$$

§7 Stokes Theorem

Definition 7.1 The *curl* of a vector field F is given by,

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Let S be an *oriented smooth* surface that is bounded by a simple, closed, smooth boundary curve C. Let F be some vector field, then,

$$\int_C F \cdot d\mathbf{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$$

This is useful when parameterizing C is difficult, unlike S. To calculate the normal vector we use the fact that $\vec{n} = \frac{\nabla g}{|\nabla g|}$.

§8 Surface Integrals

If one needs to evaluate $\int_S \vec{F} \cdot \vec{ds}$, then if we can parametrize S we obtain,

$$\int_S \vec{F} \cdot \vec{ds} = \iint_A F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right)(u,v) du dv$$

where $\times$ is the cross-product.

§9 Divergence Theorem

Used when the surface area is closed, unlike the previous methods.

Definition 9.1 The *divergence* of a vector field at a point is given by,

$$\text{div } \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

A standard example of a *divergence* 1 vector field is $\mathbf{F}(x,y,z) = \frac{1}{3}(x,y,z)$, let this be the *universal* vector field.

Remark 9.2 div curl $\mathbf{F} = 0$
Using this one can check whether a vector field $\mathbf{F}$ is the curl of another vector field $\mathbf{G}$, by simply showing that the *divergence* is non-zero.

Definition 9.3 The Laplace operator of f is given by $\Delta f = \text{div } \nabla F$

Let V be some closed region in $\mathbb{R}^3$ and let C be its boundary, then,

$$\iiint_V \text{div } \vec{F} \, dx dy dz = \int_C \vec{F} \cdot \vec{n} \text{ ds}$$

Observe that by fixing F to be the *universal* vector field we obtain that LHS is the volume of V, thus,

$$\text{vol}(V) = \frac{1}{3} \int_C (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \vec{n} \text{ ds} = \int_C x\hat{i} \cdot \vec{n} \text{ ds} = \dots$$

§10 Lagrange Multipliers Method

The method relies on the *Extreme Value Principle* (EVP) which says that a continuous functions attains both a minimum/maximum on any *compact* set A.

The LM method solves the problem of minimizing/maximizing $f(x_1, \ldots, x_n)$ given a restriction $g(x_1, \ldots, x_n) = k$ over some *compact* set A. First one must solve,

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$$

then the LM theorem says that the triple (x, y, z) is the local extremas.